18.821 SPRING 2008: POLYNOMIAL IMAGES OF CIRCLES

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Abstract. This paper considers the effect of complex polynomial maps on circles of various radii. Several phenomena are examined including cusps and windings. The results obtained for polynomial images of circles are then used to sketch an analytic proof of the Fundamental Theorem of Algebra. We go on to consider properties of curvature and their transformation under polynomials. We are able to show that the integral of curvature for an arbitrary closed curve is an integer multiple of 2π. We conclude by considering how this integral of curvature transforms under complex polynomials.

1. Introduction

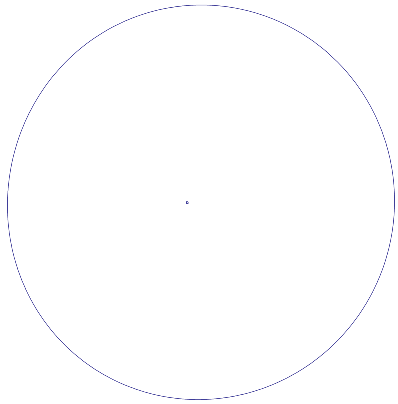
This paper examines the images of circles in the complex plane under a complex polynomial. Limiting behavior for circles of small and large radii are analyzed. In the transition between these two domains, the development of cusps, windings and local loops are observed. The roots of the complex polynomial map coincide with where the polynomial image touches the origin. The limiting behavior then suggests a proof technique for the Fundamental Theorem of Algebra, which states that every non-constant complex polynomial has a root in $\mathbb{C}$.

In order to make some of the intermediary behavior more precise, we introduce the winding number, curvature, and other notions from complex analysis and differential geometry. These concepts are used to prove that the total curvature of a general closed curve is always an integer multiple of 2π. Finally, we discuss the transformation of total curvature under a general class of complex mappings and its relation to the roots of the derivative.

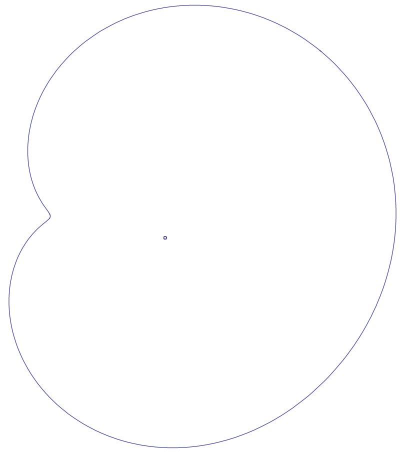
Section 2 outlines properties of polynomial images of circles. This portion of the paper discusses the cusps and windings that occur as circles of larger radii are used. Section 3 introduces concepts from complex analysis, including the definition of a winding number, the General Cauchy Formula, and the Argument Principle. Then, in Section 4, we state the Fundamental Theorem of Algebra and sketch a proof for it motivated by polynomial images of circles. Finally, we conclude the paper with Section 5 where we introduce the notion of curvature, prove that the integral of the curvature over an arbitrary closed curve is an integer multiple of 2π, and show how the integral of curvature transforms under complex polynomials.

2. Phenomenology of $p(C_R)$

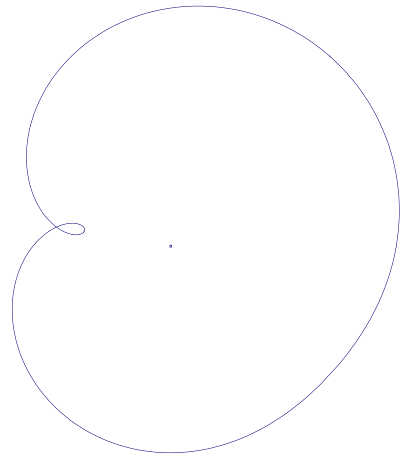
Consider the images of the circles C_R (where R denotes the radius) under a polynomial mapping $p(z) = a_n z^n + \cdots + a_0$. In this section, we describe various phenomena that we observed in the curves $p(C_R)$ when varying R. Later sections will formalize some of these observations.



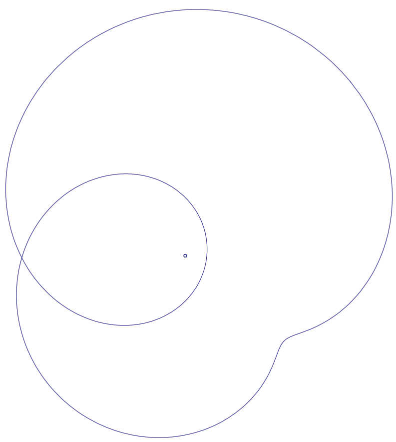
(a) $R = .1$



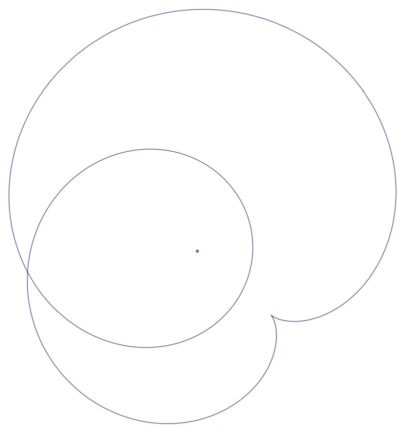
(b) $R = .4$



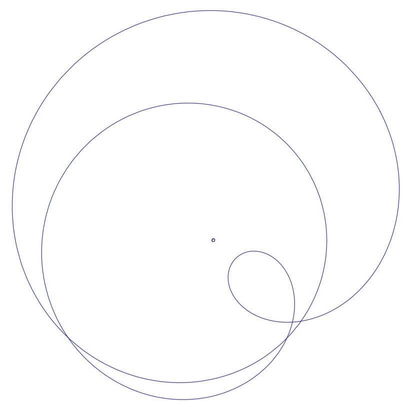
(c) $R = .5$



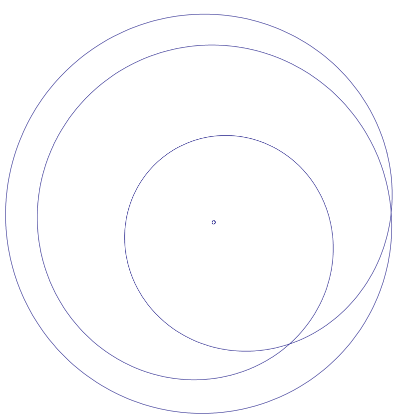
(d) $R = .9$



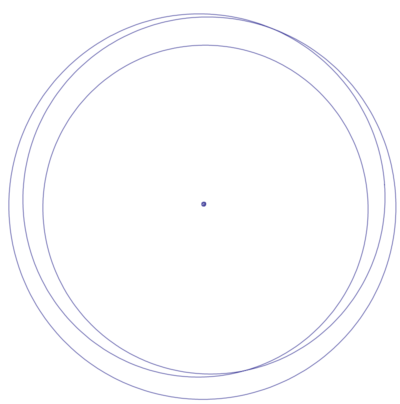
(e) $R = |\frac{1}{3}(-i + \sqrt{-4-3i})| \approx 1.07$



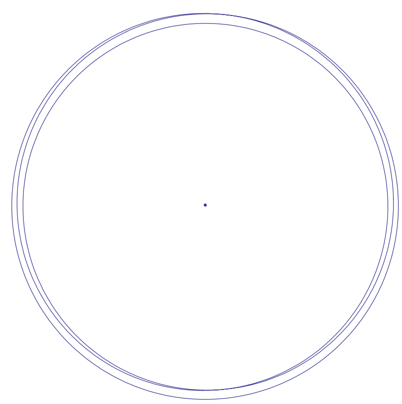
(f) $R = 1.5$



(g) $R = 3$



(h) $R = 10$



(i) $R = 30$

Figure 1: Phenomenology of $p(C_R)$ for $p(z) = z^3 + iz^2 + (1+i)z - (2+2i)$

We begin with the complex polynomial $p(z) = z^3 + iz^2 + (1+i)z - (2+2i)$. To illustrate the behavior of p, Figure 1 gives the curves $p(C_R)$ for varying R. We observe the following phenomena:

(1) The constant term of the polynomial, $a_0 = 2 + 2i$, which is indicated in each figure by a dot, lies in the interior of each of the curves.
(2) The linear term of the polynomial, $a_1 z + a_0$, mostly determines the value of the polynomial for small R and the highest order term of the polynomial, $a_n z^n$, mostly determines the value of the polynomial for large R. This change can be seen as R increases from .1 in Figure 1a to 30 in Figure 1i.
(3) Cusps form as the radius of the curve increases. Figure 1b shows the beginning of a concave deformation in the curve. The curve is still smooth, but heading toward a cusp. Figure 1c shows the curve after this cusp has passed through. Figure 1d, Figure 1e and Figure 1f show another instance of a cusp developing, but here, Figure 1e shows the exact radius of R where the cusp has formed.
(4) The image curve winds around a_0 once for very small radii and n times for very large radii. As the radius of the circle increases, cusps form and then turn into local loops. These local loops encompass a_0 for sufficiently large R, so each cusp that develops ends up adding an additional winding around a_0. This can be seen as the radius increases from 1.5 in Figure 1f to 10 in Figure 1h. The small loop that developed from the cusp of Figure 1e passes through the point a_0 and thus adds one to the number of times the curve winds around a_0. This suggests that each cusp that forms as the radius is increased adds one to the total winding number of the curve around a_0 for large R. Thus the number of windings about a_0 is one greater than the number of cusps observed as we transition from circles of small radii to circles of large radii.

These phenomena will be explored and explained in the next two sections.

3. Basic Complex Analysis

In this section we state necessary results from complex analysis to formalize the observations of Section 2. For the following discussion, see ([3] p.209-210).

A complex function $f(x+iy) = u(x,y) + iv(x,y)$ can be thought of as a function from $\mathbb{R}^2$ to itself. The derivative of $f(x,y) = (u(x,y), v(x,y))$ at a point p, given by

$$Df(p) = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \tag{3.1}$$

provides a linear approximation to f at p and so must the complex derivative. There are constraints on the derivative (3.1) known as the *Cauchy-Riemann equations*. These are the coupled differential equations

$$\begin{aligned} \partial_x u &= \partial_y v \\ \partial_x v &= -\partial_y u. \end{aligned}$$

We thus make the following definition:

Definition 3.2. A complex function $f : \mathbb{C} \to \mathbb{C}$ is **holomorphic** or **complex differentiable** on an open set $U \subseteq \mathbb{C}$ if it satisfies the Cauchy-Riemann equations and if $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists for all $z_0 \in U$.

If a function that is complex differentiable is viewed as a map from $\mathbb{R}^2 \to \mathbb{R}^2$, every partial derivative exists and is continuous. Thus, in contrast to $\mathbb{R}^2$ differentiable functions, a complex differentiable function is immediately smooth.

In Section 2, we noted the development of cusps as we varied R for $p(C_R)$. We can now define cusps as follows:

Definition 3.3. A **cusp** is a critical value of a holomorphic function $f : \mathbb{C} \to \mathbb{C}$, i.e. if z_0 is a critical point, then the cusp is $f(z_0)$.

Remark 3.4. A singular point for a complex values function is just a point $z_0 = x_0 + iy_0 \in \mathbb{C}$, where $f'(z_0) = 0$. If we treat f as map from $\mathbb{R}^2$ to itself, and put $p = (x_0, y_0)$, then $f'(z_0) = 0$ corresponds to $(\partial_x f(p), \partial_y f(p)) = 0$. This corresponds precisely to the derivative in Equation (3.1) becoming singular as a linear map at the point p. Calculating at the point p, $\partial_x f = \partial_x u + i\partial_x v = 0 \Rightarrow \partial_x u = -i\partial_x v$ and $\partial_y f = \partial_y u + i\partial_y v = 0 \Rightarrow \partial_y u = -i\partial_y v$. Thus, $\det Df(p) = \partial_x u \partial_y v - \partial_x v \partial_y u = \partial_x u \partial_y v - (i\partial_x u)(-i\partial_y v) = 0$.

It is not clear from this definition how singular points of the derivative correspond to our geometric intuition of a cusp. The following definition pins down one aspect of cusps.

Definition 3.5. A **conformal map** is a function which preserves angles.

Claim 3.6. *A holomorphic function is conformal at any point where it has non-zero derivative.*

Remark 3.7. Suppose f is holomorphic and $f'(z_0) = 0$. By Taylor's Theorem, $f(z + z_0) \approx f(z_0) + zf'(z_0) + z^2\frac{f''(z_0)}{2} + \cdots$. If z_0 is a simple zero of the derivative, then $f''(z_0) \neq 0$ and $f(z+z_0) \approx z^2$. More generally, if z_0 is a root of the derivative of order n, then $f(z+z_0) \approx z^{n+1}$. Thus the order of a critical point of f describes the local behavior of f.

We can now use Claim 3.7 to describe our geometric intuition of what cusps look like.

Remark 3.8. At every point on the curve, the tangent vector and the normal vector are necessarily at right angles. Polynomial mappings are conformal whenever the curve is not touching a critical point by Claim 3.6, so the angle between the tangent vector and the normal vector is preserved in these cases. However, if z_0 is a simple critical point, $f(z+z_0) \approx z^2$ and the angle between the tangent and normal vectors *doubles*, reflecting the observed cuspidal structure. Similarly, if z_0 is an order two critical point, the right angle between the normal and tangent vector becomes a $3 \times 90° = 270°$ angle, and so on.

Section 5 describes more features of cusps. In Section 2, local loops or *windings* were observed. In order to formalize this notion we must use the following formula from complex analysis.

Theorem 3.9 (General Cauchy Formula)**.** *([3] p.430) If $f(z)$ is holomorphic on and inside an arbitrary loop γ, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-p}dz = \nu[\gamma, p]f(p).$$

$\nu[\gamma, p]$ is the *winding number* of a curve about a point. By setting $f(z) = 1$, we can provide a formula for the winding number.

Definition 3.10. The **winding number** of a complex curve $\gamma(t)$ about a point $p \in \mathbb{C} - \gamma$ is defined as the integer value $\nu[\gamma, p] = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-p}$.

We are now able to restate a key result from complex analysis known as the *Argument Principle.*

Theorem 3.11 (The Argument Principle)**.** *If $f(z)$ is analytic inside and on a simple loop γ and N is the number of pre-images of p under f inside γ, then $N = \nu[f(\gamma), p]$.*

Thus the phenomenon of winding about a point noted in Section 2 is captured precisely in terms of the number of pre-images of that point in the interior of a curve. For certain curves we may not care *which* point is being wound around, but we might care about the more global feature of when a local loop develops in the curve. In Section 5, we will capture this notion of a local loop formally by extending the local notion of *curvature* to a global feature of *total curvature.*

4. The Fundamental Theorem of Algebra

Section 2 mentioned some observed properties of the polynomial images of circles and this section applies these observations to sketch a proof the Fundamental Theorem of Algebra, which we now state.

Theorem 4.1 (The Fundamental Theorem of Algebra)**.** *Every non-constant, single-variable polynomial with complex coefficients has at least one complex root.*

Consider the curves C_R, circles centered around the origin with radius R, and the complex polynomial p. The polynomial p is continuous, so varying R continuously deforms the image curve $p(C_R)$. By beginning with a radius R_0 such that $p(C_{R_0})$ does not wind around the origin, and increasing to radius R_1 such that $p(C_{R_1})$ does wind around the origin, the continuity of this deformation requires that there is some intermediate r such that the curve $p(C_r)$ intersects the origin. The polynomial p must have a root somewhere on this curve C_r.

To explore the details of this proof, we begin by demonstrating that there is some circle with positive radius R_0 small enough such that $p(C_{R_0})$ does not wind around the origin. As mentioned in previous sections, the images of circles $p(C_R)$ wind around the point $p(0) = a_0$, the constant term of p. Thus, for $a_0 \neq 0$ we can show that $p(C_{R_0})$ does not wind around the origin by bounding the distance $|p(z) - a_0|$. This is formalized in the following lemma.

Lemma 4.2. *Let $p(z) = \sum_{i=0}^{n} a_i z^i$ be a complex polynomial of degree n. There exists a positive $R_0 \in \mathbb{R}$, such that for $|z| = R_0$, $|p(z) - a_0| \leq \frac{1}{2}|a_0|$ and thus $p(C_{R_0})$ does not wind around the origin if $a_0 \neq 0$.*

Proof. The idea is to find a small enough R_0 to bound the non-constant terms. Specifically, take $R_0 = \frac{1}{2(1+\sum_{i=0}^{n} |a_i|)} \min\{1, |a_0|\} < 1$ and an arbitrary z such that $|z| = R_0$. Then $R_0 \geq R_0^i = |z|^i$ for $i \geq 1$. Consequently,

$$|p(z) - a_0| = \left| \sum_{i=1}^{n} a_i z^i \right| \leq \sum_{i=1}^{n} |a_i z^i| = \sum_{i=1}^{n} |a_i||z^i| \leq \sum_{i=1}^{n} |a_i| R_0 < \frac{1}{2}|a_0|$$

As this result holds for arbitrary z such that $|z| = R_0$, the first part of the claim follows.

Suppose $a_0 \neq 0$. To show that $p(C_{R_0})$ does not wind around the origin, we only need to notice that $p(C_{R_0})$ is contained in the closed ball of radius $\frac{1}{2}|a_0|$ around a_0, and this ball does not contain the origin for $a_0 \neq 0$. □

The previous lemma proves the existence of the desired R_0. To give the R_1 so that $p(C_{R_1})$ winds around the origin, we use a similar argument that relies on the dominance of the largest degree term instead of the dominance of the constant term. This argument is given in the next lemma.

Lemma 4.3. *Let $p(z) = \sum_{i=0}^{n} a_i z^i$ be a non-constant complex polynomial of degree n. There exists an $R_1 \in \mathbb{R}$ such that $|p(z) - z^n| \leq \frac{1}{2}R_1^n$ for all complex z where $|z| = R_1$. Thus, $p(C_{R_1})$ winds around the origin.*

Proof. Note that $a_n \neq 0$ and $n \geq 1$.

The idea is to find an R_1 such that the lower order terms in $p(z)$ are small relative to z^n. We can factor out the z^n term to get $|p(z) - z^n| = |z^n| \cdot \left|\sum_{i=0}^{n-1} \frac{a_i}{z^{n-i}}\right|$. The claim will be proven by bounding $|\sum_{i=0}^{n-1} \frac{a_i}{z^{n-i}}|$. This is achieved by picking a large enough R_1, specifically, $R_1 = \max\{1, 2n(\max_{0 \leq i \leq n} |a_i|)\} \geq 1$.

Take an arbitrary z such that $|z| = R_1$. Then $|z|^{n-i} = R_1^{n-i} \geq R_1$. Using that $R_1 \geq 2n|a_i|$ for any i shows

$$\left|\frac{a_i}{z^{n-i}}\right| = \frac{|a_i|}{|z|^{n-i}} \leq \frac{|a_i|}{R_1} \leq \frac{1}{2n}$$

and consequently by the triangle inequality,

$$\left|\sum_{i=0}^{n-1} \frac{a_i}{z^{n-i}}\right| \leq \sum_{i=0}^{n-1} \left|\frac{a_i}{z^{n-i}}\right| \leq \sum_{i=0}^{n-1} \frac{1}{2n} = \frac{1}{2}$$

This bounds the lower order terms. By multiplying through by z^n, we get a statement about $|p(z) - z^n|$.

$$|p(z) - z^n| = \left|\sum_{i=0}^{n-1} a_i z^i\right| = |z^n| \cdot \left|\sum_{i=0}^{n-1} \frac{a_i}{z^{n-i}}\right| \leq \frac{1}{2}|z^n| = \frac{1}{2}R_1^n$$

As this result holds for arbitrary z such that $|z| = R_1$, the claim follows. □

Remark 4.4. To see that $p(C_{R_1})$ winds around the origin, we use a "dog-walking" type argument. The idea of this argument is to notice than when a dog-owner walks around a tree and the dog is on a leash with length strictly bounded by the distance to the tree, then the dog must also walk around the tree as many times as the owner. z^n "walks" around the origin n times and "walks the dog" of $p(z)$ along with it with leash length $|p(z) - z^n|$. As $|p(z) - z^n| \leq \frac{1}{2}R_1^n$ and $|z^n| = R_1^n$, then the "dog" cannot reach the tree so it must be that $p(z)$ "walks" around, and thus winds around, the origin.

With the existence of both R_0 and R_1, we now give a sketch of a proof of the Fundamental Theorem of Algebra.

Proof sketch of the Fundamental Theorem of Algebra. There are two cases.

Case: $a_0 = 0$: Then $p(0) = 0$ and 0 is a root.

Case: $a_0 \neq 0$: Lemma 4.2 gives a positive R_0 such that $p(C_{R_0})$ does not wind around the origin and Lemma 4.3 gives an R_1 such that $p(C_{R_1})$ does wind around the origin.

As p is a polynomial it is continuous in z as a complex variable and also as a function over $\mathbb{R}^2$. The function $Re^{i\theta}$ is continuous in R and θ. Composing these continuous functions shows that the image curve $p(C_R)$ is continuous in some sense as a function of R. Thus, when taking the curves $p(C_R)$ for $R \in [R_0, R_2]$ the curve $p(C_{R_0})$ is continuously deformed into $p(C_{R_1})$. As $p(C_{R_0})$ does not wind around the origin but $p(C_{R_1})$ does, it must be that there is some $r \in [R_0, R_1]$ such that $p(C_r)$ intersects the origin.

That $p(C_r)$ intersects the origin means that there is some $\phi \in [0, 2\pi]$ such that $p(re^{i\phi}) = 0$. Then, $re^{i\phi}$ is a root of p which establishes the theorem. □

The proof is incomplete as the loose idea of continuity of curves in the deformation from $p(C_{R_0})$ to $p(C_{R_1})$ is not formal. This deformation can be seen as a version of the Intermediate Value Theorem for curves in $\mathbb{R}^2$, but this is by no means a formal proof. While we were not able to formalize this idea, the intuition is clear.

5. Curvature

Now that we have considered some of the more general behavior of polynomial images of circles, we would like to determine how the observed cusps and local loops effect curvature.

Definition 5.1. A continuously differentiable (i.e. $\mathcal{C}^1$) curve $\alpha(t)$ is **regular** if for all t $\alpha'(t) \neq 0$.

Definition 5.2. Let $\alpha : I \to \mathbb{R}^2 \cong \mathbb{C}$ be a regular $\mathcal{C}^2$ plane curve $\alpha(t) = (x(t), y(t))$. Then the **curvature** at a point x,y is defined by $\kappa = \frac{x'y''-y'x''}{(x'^2+y'^2)^{3/2}}$.

The curvature of a non-regular curve is undefined for critical points. We will assume, for convenience, that all the curves are $\mathcal{C}^2$. We now introduce the notion of *total curvature.*

Definition 5.3. The **total curvature** of a smooth plane curve γ is $K(\gamma) := \int_\gamma \kappa ds$.

Example 5.4. Suppose $\alpha(t) = Re^{2\pi it} \cong R(\cos 2\pi t, \sin 2\pi t)$, $t \in [0, 1]$ is a circle with radius R. Then $\alpha'(t) = 2\pi i Re^{2\pi it} \cong 2\pi R(-\sin 2\pi t, \cos 2\pi t)$ and $\alpha''(t) = -4\pi^2 Re^{2\pi it} \cong -4\pi^2 R(\cos 2\pi t, \sin 2\pi t)$. Applying Definition 5.2 we calculate

$$\begin{aligned}\kappa &= \frac{(-2\pi R\sin 2\pi t)(-4\pi^2 R\sin 2\pi t) - (-4\pi^2 R\cos 2\pi t)(2\pi R\cos 2\pi t)}{[(-2\pi R\sin 2\pi t)^2 + (2\pi R\cos 2\pi t)^2]^{3/2}} \\ &= \frac{8\pi^3 R^2}{(2\pi R)^3} \\ &= \frac{1}{R}\end{aligned}$$

Applying Definition 5.3, we see that $K(\alpha) = \int_\alpha \kappa ds = \int_0^1 \frac{1}{R}\sqrt{x'^2 + y'^2}dt = \int_0^1 \frac{2\pi R}{R}dt = 2\pi$.

Despite a potential dependency on the radius of the circle, the total curvature of any positively oriented circle is a constant 2π. We would like to see what other curves have such well-behaved total curvatures and just how much we can change a curve without effecting its total curvature.

Definition 5.5. Suppose $\alpha(t) = (x(t), y(t))$ is a regular closed plane curve and $\alpha'(t) = (x'(t), y'(t))$ is the tangent vector to α at t, after translation. The integer number I of (signed) complete revolutions that the tangent vector makes around the origin is the **rotation index** of α [2].

Note that I can be negative if the curve is negatively oriented and the tangent vector rotates around the origin in a clockwise direction. Consequently, a tangent vector who rotates once positively around the origin and once negatively as rotation index $1 - 1 = 0$. The infinity symbol is a good example of this possibility.

Claim 5.6. *Suppose α is a closed plane curve with rotation index I. Then the total curvature $K(\alpha) = 2\pi I$.*

Proof. The following argument only provides heuristic evidence for the claim. Locally we may define the angle that α' makes with the x-axis by $\theta(t) = \tan^{-1}(\frac{y'(t)}{x'(t)})$. We can then write locally the tangent vector as a function of θ so $\alpha'(t) = (x'(t), y'(t)) = (\cos\theta(t), \sin\theta(t))$. Differentiating, we find that

$$\theta' = [\tan^{-1}\frac{y'}{x'}]' = \frac{x'y'' - y'x''}{(x')^2}\frac{1}{1 + (y'/x')^2} = \frac{x'y'' - y'x''}{x'^2 + y'^2}.$$

Our assumption that $\alpha(s) = (x(s), y(s))$ is $\mathcal{C}^2$ implies that θ' is continuous and thus integrable. In particular,

$$\begin{aligned}
\int_\alpha \kappa ds &= \int_0^1 \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}} ds \\
&= \int_0^1 \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}} \sqrt{x'^2 + y'^2} dt \\
&= \int_0^1 \frac{x'y'' - y'x''}{x'^2 + y'^2} dt \\
&= \int_0^1 \theta' dt \\
&= \theta(1) - \theta(0)
\end{aligned}$$

Intuitively, $\theta(1) - \theta(0)$ captures the number of times the tangent vector rotates around the origin. By Definition 5.5, $\theta(1) - \theta(0) = 2\pi I$. □

If we identify $\mathbb{R}^2 \cong \mathbb{C}$ and appeal to complex analysis terminology, we recognize *the rotation index of the tangent vector is the winding number of the derivative.* We now have an intuitive reason to believe that for smooth closed curves, the total curvature is just 2π times the winding number of the derivative about the origin. This is captured in the following theorem.

Theorem 5.7. *Let $\gamma(t) = x(t) + iy(t)$ be a closed smooth complex curve. The total curvature of γ about the origin is $2\pi\nu[\gamma', 0]$.*

Proof. The winding number of $\gamma' = x'(t) + iy'(t)$ around the origin is given by $\frac{1}{2\pi i}\int_{\gamma'} \frac{dz}{z}$, which must be an integer by Theorem 3.9, the General Cauchy Formula. We thus calculate that:

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma'}\frac{dz}{z} &= \frac{1}{2\pi i}\int_0^1 \frac{\gamma''(t)}{\gamma'(t)}dt \\
&= \frac{1}{2\pi i}\int_0^1 \frac{x''+iy''}{x'+iy'}\frac{x'-iy'}{x'-iy'}dt \\
&= \frac{1}{2\pi i}\int_0^1 \frac{x''x'-iy'x''+iy''x'+y''y'}{x'^2+y'^2}dt \\
&= \frac{1}{2\pi i}\int_0^1 \frac{x''x'+y''y'}{x'2+y'^2}dt + \frac{1}{2\pi}\int_0^1 \frac{y''x'-y'x''}{x'^2+y'^2}dt \\
&= \frac{1}{2\pi}\int_0^1 \kappa ds.
\end{aligned}$$

In the next to last line, we know the left integral must be zero because the integral is complex and the winding number is real (more specifically, an integer). The last line thus follows because the right integral is exactly $\int \theta' dt = \int \kappa ds$. □

Corollary 5.8. *The total curvature K of a closed smooth path is an integer multiple of 2π.*

Proof. Suppose the winding number of the derivative of a curve is $n \in \mathbb{Z}$. By Theorem 5.7 $n = \frac{1}{2\pi}\int \kappa ds$ and $\int \kappa ds = 2\pi n$ and we obtain the desired result. □

We can now consider more general mappings of circles and their effect on curvature.

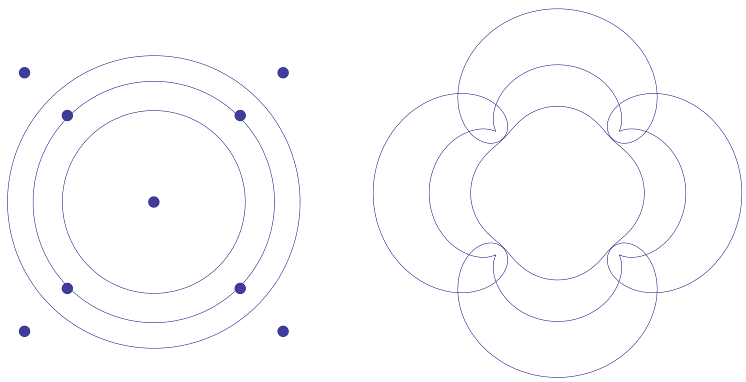
Lemma 5.9. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function and γ is a simple closed curve, then $\nu[[f\circ\gamma]', 0] = \nu[f'\circ\gamma, 0] + \nu[\gamma', 0]$.*

Proof. The lemma follows from the following calculation.

$$\begin{aligned}
\nu[[f\circ\gamma]', 0] &= \frac{1}{2\pi i}\int_{[f\circ\gamma]'}\frac{dz}{z} \\
&= \frac{1}{2\pi i}\int_0^1 \frac{d([f\circ\gamma]')}{[f\circ\gamma]'}dt \\
&= \frac{1}{2\pi i}\int_0^1 \frac{d(f'(\gamma(t))\gamma'(t))}{f'(\gamma(t))\gamma'(t)}dt \\
&= \frac{1}{2\pi i}\int_0^1 \frac{f''(\gamma(t))\gamma'^2(t)+f'(\gamma(t))\gamma''(t)}{f'(\gamma(t))\gamma'(t)}dt \\
&= \frac{1}{2\pi i}\int_0^1 \frac{f''(\gamma(t))\gamma'(t)}{f'(\gamma(t))}dt + \frac{1}{2\pi i}\int_0^1 \frac{\gamma''(t)}{\gamma'(t)}dt \\
&= \frac{1}{2\pi i}\int_{f'(\gamma)}\frac{dz}{z} + \frac{1}{2\pi i}\int_{\gamma'}\frac{dz}{z}
\end{aligned}$$

This last line is exactly $\nu[f'\circ\gamma, 0] + \nu[\gamma', 0]$. □

Theorem 5.10. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function and γ is a simple closed curve that winds once around $\{p_1, \ldots, p_N\}$ where $f'(p_i) = 0$ are roots of the derivative. Then the total curvature of the image curve is $K(f(\gamma)) = \pm(2\pi N + 2\pi)$.*



(a) $C_{.5}, C_{.66874}, C_{.8}$, Roots of f, f' (b) $f(C_{.5}), f(C_{.66874}), f(C_{.8})$

Figure 2: Roots of $z^5 + z$, $5z^4 + 1$

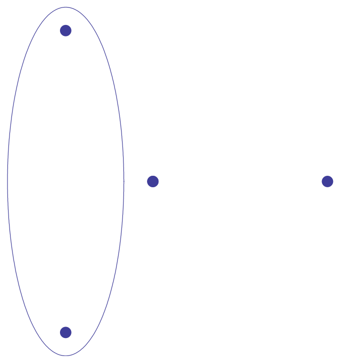
Proof. By Theorem 5.7 the total curvature of the image curve $f \circ \gamma$ is just $K(f(\gamma)) = 2\pi\nu[[f \circ \gamma]', 0]$. By Lemma 5.9 this is just 2π times the sum of the winding numbers $\nu[f' \circ \gamma, 0] + \nu[\gamma', 0]$. By the Argument Principle (Theorem 3.11) $\nu[f' \circ \gamma, 0]$ is just the number of roots of f' that γ encloses. By hypothesis, this is N. If γ is positively-oriented and thus winds around the roots in a counter-clockwise manner then $K(f(\gamma)) = 2\pi N + 2\pi$, and this quantity is negative if γ is negatively-oriented and f' is orientation-preserving. □

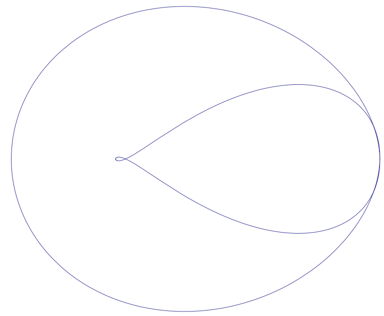
Remark 5.11. We would like to generalize the above result to include more general curves and their images, but the Argument Principle, as stated, only provides information about *simple* closed curves. There should be no problems with curves that have self-intersections and wind around roots of the derivative multiple times. Regardless, the above result illustrates the most important features of the theory. In particular, it should be noted that the above result works for *any holomorphic function*. Polynomials happen to always be holomorphic functions and from the previous discussion regarding the Fundamental Theorem of Algebra, they are guaranteed to have roots.

6. Numerical Investigations

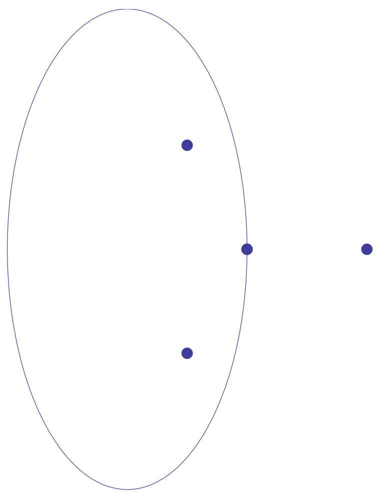
It should be noted that when the problem of curvature of polynomial images of circles is first treated computationally, it is easy to be misled. It is easy to show that a circle, whose total curvature is 2π, when mapped via a degree n polynomial such as z^n, has total curvature of $n2\pi$. In particular, for circles large enough this is true for any degree n polynomial. One might mistakenly conjecture a different version of Theorem 5.10 that says $K(f(\gamma)) = n2\pi$. However, in Figure 2, we consider the mapping $f(z) = z^5 + z$ and various circles in relation to the roots of the function and its derivative. The roots of f are simply $\{0, -(-1)^{1/4}, (-1)^{1/4}, -(-1)^{3/4}, (-1)^{3/4}\}$ and the roots of f' occur at $\left\{-\left(-\frac{1}{5}\right)^{1/4}, \left(-\frac{1}{5}\right)^{1/4}, -\frac{(-1)^{3/4}}{5^{1/4}}, \frac{(-1)^{3/4}}{5^{1/4}}\right\}$, which all have norm approximately equal to 0.66874.

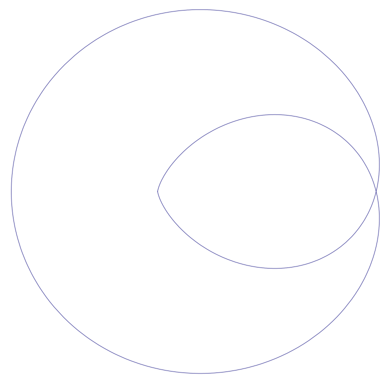
The total curvature of the curves $f(C_{.5}), f(C_{.66874}), f(C_{.8})$ are, respectively $K = \{2\pi, 6\pi, 10\pi\}$. The value for $K(f(C_{.66874})) = 6\pi$ can in some ways be considered a numerical fiction since the curvature at these cusp points is undefined. Mathematica consistently assigns $k\pi$ to the curvature for each cusp, where k is the order of the zero of the derivative.



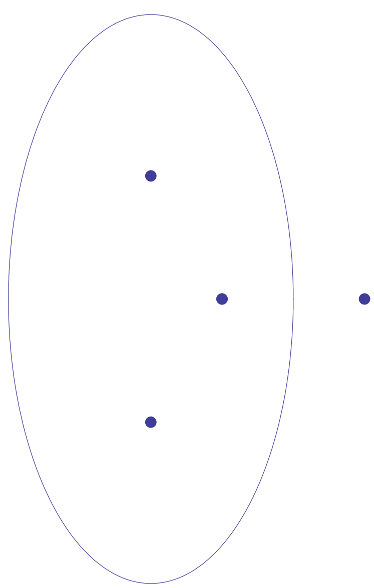


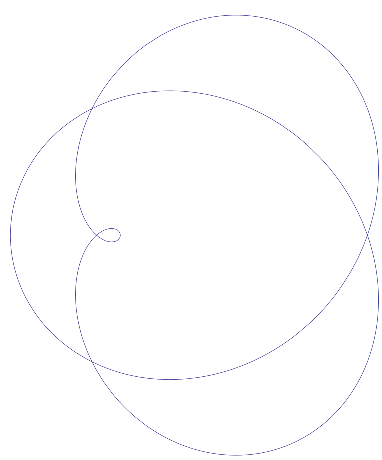
(a) Two Roots of f: Domain (b) Two Roots of f: Image





(c) Two Roots of f, f': Domain (d) Two Roots of f, f': Image





(e) Two Roots of f, f': Domain (f) Two Roots of f, f': Image

Figure 3: Roots of $(z-2)^3 - 1$ and its Derivative

As a final experiment, we consider winding around roots of the function without winding around roots of the derivative. The Argument Principle tells us that we must wind around the origin once for every root of the function, but it is not clear how we wind around the origin additional times without without effecting the total curvature. In Figure 3, we consider such a situation. If one considers the function $f(z) = (z-2)^3 - 1$, which has roots $\left\{3, \frac{1}{2}\left(3 - i\sqrt{3}\right), \frac{1}{2}\left(3 + i\sqrt{3}\right)\right\}$, and whose derivative has a root of order 2 at 2, we are able to selectively wind around roots using ellipses.

The winding number about the origin of the image curve in Figure 3b is indeed two, but the curve performs a twist near the center of the image so the rotation index of the tangent vector is one. Numerical calculation confirms that the total curvature is still 2π for Figure 3b. The total curvature for Figure 3d is $2\pi + 2\pi = 4\pi$, reflecting that this is a cusp of order two. Finally, the total curvature for Figure 3f is $2\pi + 2 \times 2\pi = 6\pi$, reflecting the result of Theorem 5.10.

7. Division of Labor

Justin Curry - Introduction, Complex Analysis Revisited, Curvature
Michael Forbes - Phenomenology of $p(C_R)$, The Fundamental Theorem of Algebra
Matthew Gordon - Editing

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