18.821 SPRING 2008: COLORING KNOTS

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ABSTRACT. In this paper, we discuss colorability of knot-projections. In particular, we prove that colorability is a knot-invariant over finite abelian groups and that we can use this to show that knots are inequivalent. We also demonstrate that we retain much of our ability to distinguish knots even if we only color over the prime fields.

1. INTRODUCTION

A knot is a smooth embedding of a circle into $\mathbb{R}^3$. For the purposes of this paper, a knot can simply be thought of as a piece of string that has been wound around itself and then had its ends joined together. Knots are equivalent if they can be deformed into each other without tearing. More formally, two knots are defined to be equivalent if there exists an orientation-preserving homeomorphism between them.

It is convenient to project knots onto $\mathbb{R}^2$ to make them easier to work with. We can obtain a knot-projection of any knot by mapping each point (x, y, z) in the knot to the point (x, y) in the knot-projection.

To extend the notion of knot equivalence to $\mathbb{R}^2$, Kurt Reidemeister proposed three transformations on knot-projections which are known as the Reidemeister moves. Reidemeister was able to prove that knot-projections of equivalent knots are related by a finite sequence

Figure 1: Reidemeister move I [2]

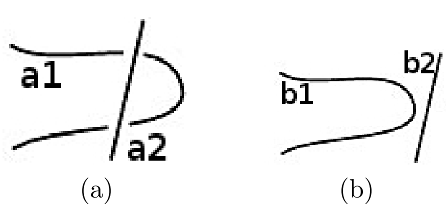
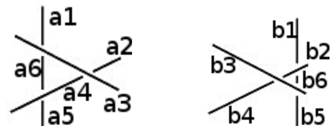


Figure 2: Reidemeister move III [2]



(a) (b)

Figure 3: Reidemeister move III [2]

of these three moves. We can thus repeatedly apply the Reidemeister moves to enumerate all knot-projections that are equivalent to an initial knot-projection.

Figure 4 demonstrates the process of using Reidemeister moves to move between equivalent knot-projections. In this case, we begin with the unknot and apply the first Reidemeister move twice.

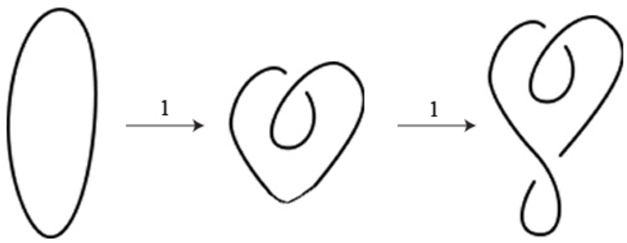


Figure 4: Applying Reidemeister Moves To The Unknot [3]

While it might seem that any knot-projection that is equivalent to the unknot will be quite simple, Figure 5 shows that this is not the case.

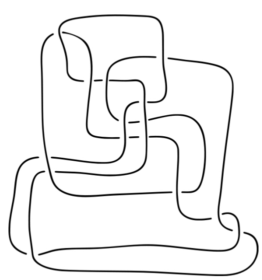


Figure 5: A Complex Unknot [3]

The Reidemeister moves give a way to prove that two knots are equivalent but they do not give a way to prove that two knots are inequivalent. For this task, we utilize knot-invariants. A knot-invariant is a property of a knot that is shared by all equivalent knots. Knot-invariants do not necessarily take different values for knots that are inequivalent, however, so we cannot use knot-invariants to prove that knots are equivalent.

In the remainder of the paper, we prove that colorability is a knot-invariant over nite abelian groups and that we can use this to show that knots are inequivalent. In Section 2, we introduce colorability and illustrate how to use 3-colorability to prove that two knots are inequivalent. In Section 3, we show how we can represent knot-projections as systems of homogeneous linear equations where colorings of the knot-projections correspond to solutions to the systems of equations. We also demonstrate the limits of using 3-colorability in proving that knots are inequivalent. In Section 4, we introduce the connected sum operation as a tool for generating new knots and explore the effect that the operation has on the linear systems we have developed. In Section 5, we generalize coloring to groups, present the proof that it is a knot-invariant for finite abelian groups, and show that this generalized formulation of coloring allows us to overcome the limitations of 3-colorability. Finally, in Section 6, we demonstrate that we retain much of our ability to distinguish knots even if we only color over the prime fields.

2. Colorability

In Section 1, we discussed how we could use a knot-invariant to prove that two knots are inequivalent. In this paper, we examine how to use colorability as our knot-invariant. Colorability is shown to be a knot-invariant in Corollary 5.8, so we use this result without justification for now.

Before defining colorability formally, let us introduce some basic terminology:

Definition 2.1. A **crossing** occurs in a knot-projection when the points $p_1 = (x, y, z_1)$ and $p_2 = (x, y, z_2)$ in the knot, where $z_1 \neq z_2$, map to the same point (x, y) in the knot-projection.

We call p_1 the over-crossing and p_2 the under-crossing where $z_1 > z_2$. It is not necessary to deal with cases where more than two points in the knot have the same (x, y) because infinitesimal movements can be used to move the knot so that this is not the case.

Definition 2.2. In a knot-projection K, an **arc** is a maximal path not containing any under-crossings. The set of arcs in K is denoted $A(K)$.

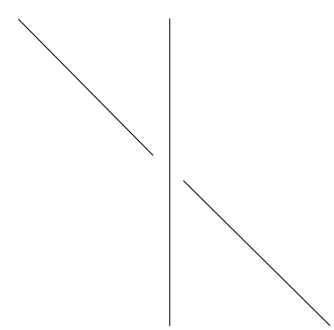


Figure 6: A Crossing

Three (not necessarily distinct) arcs meet at each crossing in a knot-projection. There is the arc that passes through the over-crossing as well as the arc that is split into two arcs by the under-crossing. For a given crossing, we call the arc that passes through the over-crossing an **over-arc** and the two arcs terminating at the under-crossing **under-arcs**.

Figure 6 shows our convention for drawing crossings, which has been chosen to emphasize that the under-crossing gives us an arc on either side of the over-arc.

Definition 2.3. A **coloring** of a knot-projection K is a function that assigns one of a set of colors to each member of $A(K)$ such that the three arcs of the knot-projection that meet at each crossing in K either all have the same assignment or all have different assignments.

We refer to this constraint on the arcs that meet at a crossing as the **crossing condition**. A **trivial coloring** of a knot-projection is a coloring where all arcs take the same color. When we're coloring over a set of n colors, this means there will be n trivial colorings. If there is also a non-trivial coloring, then we call that knot-projection n-**colorable**.

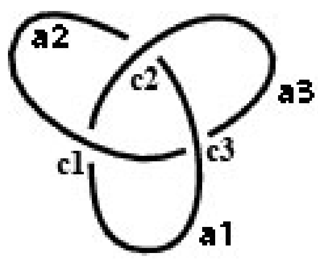


Figure 7: Labeled Trefoil Knot [2]

We can now use the notion of 3-colorability to prove that two knots are inequivalent. Consider the unknot, which is the knot we started with in Figure 4, and the trefoil knot, which is pictured in Figure 7. The unknot can only have trivial colorings because there is only one arc in the entire knot for us to assign a color. The trefoil knot, however, has three arcs and three crossings. If we let each of the arcs be a different color, we find that the crossing conditions are met for each of the crossings. Therefore the trefoil knot is 3-colorable and the unknot is not 3-colorable. This means that these two knots are inequivalent since coloring is a knot-invariant.

3. Representing Knot Projections with Linear Systems

In this section, we show how the coloring of a knot with three colors can be reduced to solving a system of linear equations over a field of three elements, with the identification 0 = red, 1 = green, and 2 = blue. By using a numerical assignment we can reduce the crossing conditions for a valid coloring into a system of equations. We thereby turn a combinatorial question of coloring into a problem of linear algebra.

Proposition 3.1. *Let K be a knot-projection. Enumerate the arcs $\{a_1, \dots, a_n\}$ and the crossings $\{c_1, \dots, c_k\}$. At each crossing c_ℓ the crossing condition requires that: (1) $a_i = a_j = a_k$ or (2) $a_i \neq a_j \neq a_k$ and $a_i \neq a_k$. These reduce to the single equation*

$$a_i + a_j + a_k \equiv 0 \bmod 3 = 0_{\mathbb{Z}_3}. \tag{3.2}$$

Proof. One can see that condition (1) obeys equation (3.2) since $a_i + a_j + a_k = 3a_i = 0_{\mathbb{Z}_3}$. Similarly, (2) gives rise to permutations of the equation $a_i + a_j + a_k = 0 + 1 + 2 = 0_{\mathbb{Z}_3}$. It can be verified that all other possible assignments that violate the definition of a coloring fails equation (3.2) as well. □

For now we will assume that $0 = 0_{\mathbb{Z}_3}$. Proposition 3.1 gives a general recipe for associating to any knot projection a linear system of equations. The following result tells us this is always a *square* system.

Lemma 3.3. *The number of arcs equals the number of crossings, so $n = k$ in Proposition 3.1.*

Proof. By following a knot in one direction in three dimensions we give it an orientation. This orientation of the knot induces an orientation on the knot-projection. At each crossing there is exactly one arc that terminates. We create a bijection by assigning every arc that ends at a crossing to that crossing. $\square$

Definition 3.4. Let K be a knot-projection with crossings $c_1, \dots, c_n$ and arcs $a_1, \dots, a_n$. Suppose that at each c_ℓ an equation of form (3.2) holds. The **coloring matrix** of K is an $n \times n$ matrix M_K over $\mathbb{Z}_3$ defined by $\sum_{1 \le \ell \le n} (e_{\ell,i(\ell)} + e_{\ell,j(\ell)} + e_{\ell,k(\ell)})$, with $e_{r,s}$ the matrix unit.

Proposition 3.5. *For a knot-projection K with n arcs, let $\vec{x} \in \mathbb{Z}_3^n$, then $\vec{x}$ is a coloring if and only if $\vec{x} \in \ker(M_K)$ where M_K is the coloring matrix.*

Proof. By construction, a coloring is a solution to the homogeneous system of equations in Proposition 3.1. This is equivalent to finding vectors in the null-space of the associated coloring matrix M_K. $\square$

Corollary 3.6. *A knot-projection K is 3-colorable if* $\dim[\ker(M_K)] > 1$.

Proof. The trivial coloring corresponds to any vector in the span of $\vec{x} = (1, \dots, 1)^T$ with length $n = |A(K)|$. Any non-trivial coloring is not in the span of this vector but by the above proposition lies in the kernel. $\square$

We will show later in the paper that the dimension of the kernel is a knot-invariant. By applying Corollary 3.6 we will establish that 3-colorability is a knot-invariant.

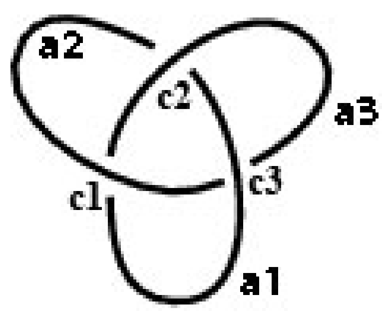


Figure 8: The Trefoil Knot. Edited from Wolfram's Mathworld.

Example 3.7. We begin with our first knot that presents a non-trivial coloring – the trefoil knot. Figure 8, shows that the arcs have been labeled $\{a_1, a_2, a_3\}$ and the three crossings labeled $\{c_1, c_2, c_3\}$. Proposition 3.1 requires that at each crossing c_i the equation $a_1 + a_2 + a_3 = 0$ must hold. We can write this more compactly with the coloring matrix as

$$M_K \vec{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \vec{0}, \tag{3.8}$$

where the ith row of M_K corresponds to the crossing condition at c_i. We observe that $\ker(M_K)$ has the following basis: $\vec{x}_1 = (1,1,1)^T$ and $\vec{x}_2 = (0,1,2)^T$. $\vec{x}_1$ corresponds to the trivial coloring, and $\vec{x}_2$ corresponds to a non-trivial coloring. It is important to notice that since we are working over $\mathbb{Z}_3$ the span of $\vec{x}_1$ includes all possible choices for the trivial coloring. Similarly, $\text{span}(\vec{x}_1, \vec{x}_2) - \text{span}(\vec{x}_1) = \{\text{the set of non-trivial colorings}\}$ and we account for the different ways in which the trefoil might be colored with three colors.

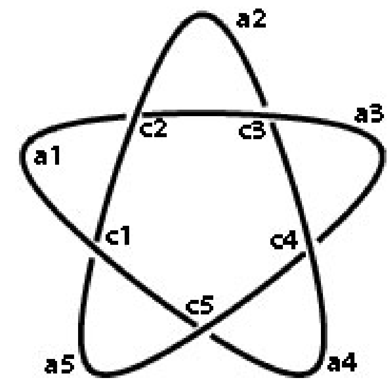


Figure 9: The 5-Foil. Edited from Wolfram's Mathworld.

Example 3.9. A slightly more complicated example illustrates that the notion of 3-colorability fails to distinguish certain inequivalent knots. Figure 9 depicts the projection of the **5-foil** (or more commonly known as **Solomon's seal knot**). The recipe of Proposition 3.1 provides the following coloring matrix:

$$M_{K_5}\vec{x} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \vec{0}. \tag{3.10}$$

Let $\vec{c}_i$ denote the ith row of M_{K_5} (we make this choice to emphasize that each row corresponds to the arcs at a specific crossing). It is clear that $\vec{c}_2, \vec{c}_3, \vec{c}_4$ are all independent. We want to argue by contradiction that $\vec{c}_5$ is independent as well. If $\vec{c}_5 \in \text{span}(\vec{c}_2, \vec{c}_3, \vec{c}_4)$ then $\vec{c}_2$ and $\vec{c}_4$ must each have 1 as their coefficients since the first and fifth entry in $\vec{c}_5$ is 1. Similarly the third entry of $\vec{c}_5$ is 0, which requires that the coefficient of $\vec{c}_3$ is also 1, but this would imply that $c_{22} + c_{32} + c_{42} = 1 + 1 + 0 = 2 \neq c_{52}$ where c_{i2} denotes the 2nd entry of the ith row. Thus $\vec{c}_5 \notin \text{span}(\vec{c}_2, \vec{c}_3, \vec{c}_4)$. The Rank-Nullity Theorem states that

$$\dim[\text{Im}(M_K)] + \dim[\text{Ker}(M_K)] = n,$$

where M_K is any $n \times n$ coloring matrix. Since $\{\vec{c}_2, \vec{c}_3, \vec{c}_4, \vec{c}_5\}$ are independent vectors in the image of M_5 we have that $\dim\ker(M_5) = 5 - 4 = 1$ and thus by Corollary 3.6, the 5-foil is *not 3-colorable*.

4. Coloring the Connected Sum

Given two knots K_1 and K_2, we can form a new knot $K = K_1 \# K_2$ known as the **connected sum** of K_1 and K_2. The connected sum is constructed by making a cut in the both knots and then gluing the ends of K_1 with K_2. By projecting onto the plane, we can preform the connected sum on two knot-projections. The connected sum operation for two trefoil knots is illustrated in Figure 10.

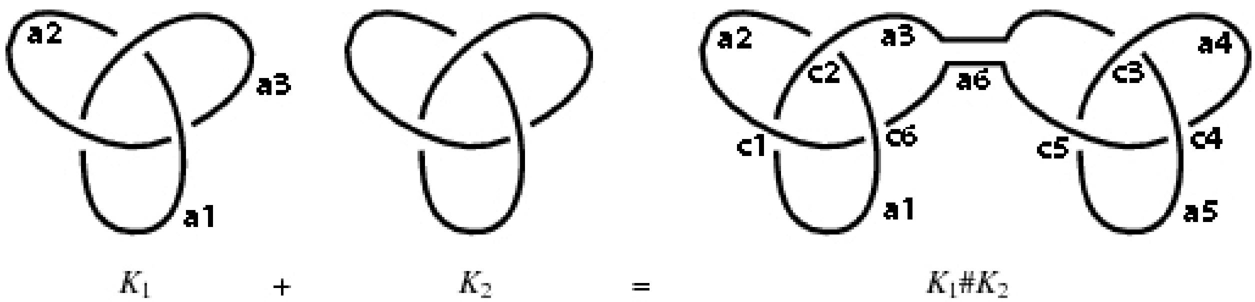


Figure 10: Connected Sum of Two Trefoil Knots. Edited from Wolfram's Mathworld.

Example 4.1. Following the same procedure outlined in Proposition 3.1, we create the associated coloring matrix for the connected sum of two trefoils depicted in Figure 10.

$$M_K\vec{x} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \vec{0}. \tag{4.2}$$

Once again letting $\vec{c}_i$ denote the ith row, we see that $\{\vec{c}_1, \vec{c}_3, \vec{c}_4\}$ are all independent. By the Rank-Nullity Theorem $\dim[\ker(M_K)] = 6 - 3 = 3$. By inspection, we find a basis for the kernel:

$$\begin{aligned} \vec{x}_1 &= (1,1,1,1,1,1)^T \\ \vec{x}_2 &= (0,1,2,0,1,2)^T \\ \vec{x}_3 &= (1,1,1,2,0,1)^T \end{aligned}$$

We see that $\vec{x}_1$ corresponds to the trivial coloring and that $\vec{x}_2, \vec{x}_3$ correspond to two linearly independent non-trivial colorings. We can interpret $\vec{x}_2$ as giving K_1 and K_2 a non-trivial coloring and then gluing arcs that agree on a given color. Similarly, $\vec{x}_3$ corresponds to giving K_2 a non-trivial coloring and then coloring K_1 so that its entire color agrees with the attaching arc from K_2.

The fact that the connected sum of two trefoils has $\dim[\text{Ker}(M_K)] = 3$ will be very important once we establish that the dimension of the kernel and thus the number of colorings is a knot invariant. We can then distinguish the connected sum of two trefoils from a trefoil itself. By considering the dimension of the kernel of the connected sum, we can prove when the connected sum is inequivalent to one of its constituent knots.

Proposition 4.3. *Fix two knot projections K_1 and K_2 and their associated coloring matrices. Suppose* $\dim[\mathrm{Ker}(M_{K_i})] = C_i$ *for* $i = 1, 2$, *then* $C_{K_1\#K_2} = \dim[\mathrm{Ker}(M_{K_1\#K_2})] \geq C_1 + C_2 - 1$.

Proof. Given any coloring of K_1 we can choose a trivial coloring of K_2 so that the color of the attaching arcs agree. This establishes that the connected sum admits at least C_1 independent colorings. Similarly, given any coloring of K_2 we can fix a trivial coloring of K_1 so that the arcs agree in color. Since we have already counted the trivial coloring for the entire knot once in C_1 we must subtract one from C_2 to avoid double counting. Since any trivial coloring is independent of a non-trivial coloring, this process will only count independent colorings and thus we have established a lower bound on the dimension of the kernel. □

Remark 4.4. Suppose that K_1 and K_2 are two knot projections. Suppose that K_2 is 3-colorable, then $C_2 \geq 2$. Thus $C_{K_1\#K_2} \geq C_1 + 2 - 1 = C_1 + 1 > C_1$. We then have that K_1 and $K_1\#K_2$ must differ. In particular, since the trefoil is 3-colorable and by applying the connected sum operation repeatedly, we obtain an infinite number of inequivalent knots.

By associating knot projections with matrices we can use the connected sum operation to induce an operation on matrices without making reference to their associated knots. This operation is embodied in the following Theorem.

Theorem 4.5. *If M_{K_1} is an $n \times n$ coloring matrix and M_{K_2} is a $k \times k$, both constructed in a suitable fashion, then the connected sum operation corresponds to the following operation on the augmented $(n+k) \times (n+k)$ coloring matrix:*

$$\left[\begin{array}{c|c} M_{K_1} & \mathcal{O}_{n\times k} \\ \hline \mathcal{O}_{k\times n} & M_{K_2} \end{array}\right] = \left[\begin{array}{ccc|ccc} & M'_{K_1} & & & \mathcal{O}_{n-1\times k} & \\ \hline 1 & * & 1 & 0 & \cdots & 0 \\ \hline & \mathcal{O}_{k-1\times n} & & & M'_{K_2} & \\ \hline 0 & \cdots & 0 & 1 & *' & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} & M'_{K_1} & & & \mathcal{O}_{n-1\times k} & \\ \hline 0 & \cdots & 1 & 1 & *' & 0 \\ \hline & \mathcal{O}_{k-1\times n} & & & M'_{K_2} & \\ \hline 1 & * & 0 & 0 & \cdots & 1 \end{array}\right].$$

Where M' refers to the restriction of M to its first n or k rows respectively.

Proof. In order for the matrices to be 'suitable' we require that the matrices M_{K_1} and M_{K_2} be constructed in the following nice way. Give each knot projection an orientation and label each arc so that at each crossing the next arc is the under-arc that follows the orientation. Label each crossing so that arc a_i terminates at c_i. Once we have constructed the matrices in this manner, we must glue the final arcs a_n of K_1 and b_k of K_2 so that the orientations on both knots agree. Gluing that respects the orientations then provides the assignment $a_{n+i} := b_i$ for $1 \leq i \leq k$ in the new connected sum knot. We further require the two knots be joined at a point so that a_n does not participate in any other arcs before terminating at c_n and that b_k not participate in any other arcs before ending at c'_k. By gluing in this fashion, we preserve all the relations between arcs in the first $n-1$ crossings in K_1 and the first $k-1$ crossings in K_2. This means that only the nth and $(n+k)$th rows are changed in the associated matrix. Suppose before the connected sum operation, a_n terminated at c_n with over-arc a_j and following under-arc a_1 (by construction). Similarly, let b_k end at over-arc $b_{j'}$ and under-arc b_1. After gluing, a_n ends at over-arc $b_{j'} =: a_{n+j'}$ and under-arc $b_1 =: a_{n+1}$, while $b_k =: a_{n+k}$ ends at over-arc a_j and under-arc a_1. □

5. Generalizations of Coloring to Groups

Previous sections explored coloring knot-projections in a combinatorial way when coloring over sets such as {red,green, blue} and also within a linear algebra framework when coloring over the field $\mathbb{Z}_3$. We claimed that over this field, the number of colorings of a knot-projection is a knot-invariant. This invariant distinguishes the trefoil knot from the unknot but fails to distinguish the 5-foil knot from the unknot. In this section, coloring is generalized to groups and the invariance of coloring over certain groups shows the 5-foil differs from the unknot.

Defining coloring over arbitrary sets can be done in several ways. A natural method returns to the combinatorial definition of the crossing condition: the colors of the arcs in any crossing must all be different or must all be the same. Axioms can be developed to give this relation knot-invariance. When the set of colors has a group structure, these axioms give a notion of coloring over arbitrary groups. We will not take this approach as it seems too general for some results in Section 6, namely Lemma 6.1 and its consequences such as Theorem 6.2.

We avoid the axiomatic approach to avoid non-abelian groups and so work exclusively with abelian groups. The main motivation for abelian groups is to allow the equations to respect the symmetry of the under-arcs at any crossing. Our generalization is in the spirit of the linear systems of Section 3. That framework uses equations with integer coefficients and such equations can be naturally interpreted over an abelian group structure as equations with repeated addition.

With the abelian group structure, a new linear equation may be needed to define the crossing condition. For a crossing with over-arc a_i and under-arcs a_j,a_k, the condition could be as general as $nf(a_i) + mf(a_j) + kf(a_k) = 0$ for $n, m, l \in \mathbb{Z}$ and f a map from the arcs of the knot-projection to the elements of an abelian group. However, the symmetry of the under-arcs requires that $m = l$. For simplicity, we take $m = l = 1$. n is determined by the requirement that the crossing condition enforces knot-invariance of colorings over the Reidemeister moves. In the first Reidemeister move from Figure 1, it must be that Figure 1b can replace Figure 1a seamlessly, meaning that in Figure 1a $f(a_1) = f(a_2)$. Combining with the crossing condition $nf(a_1) + f(a_1) + f(a_2) = 0$ determines that $n = -2$. Therefore, the new crossing equation is $-2f(a_i) + f(a_j) + f(a_k) = 0$.

We can now state some basic definitions and results when colorings are generalized to abelian groups.

Definition 5.1. A **coloring** of a knot-projection K over an abelian group G is a function $f : A(K) \to G$ such that, at a crossing of over-arc $a_i \in A(K)$ and under-arcs $a_j, a_k \in A(K)$, $2f(a_i) = f(a_j) + f(a_k)$. The **set of colorings** is denoted $C_G(K)$.

Remark 5.2. The notions of a **trivial coloring** and the **coloring matrix** M_K of a knot-projection can be inherited from previous sections with the natural re-definitions.

Remark 5.3. Over $\mathbb{Z}_3$, the generalized notion of coloring has the crossing equation $f(a_i) + f(a_j) + f(a_k) = 0$ as $-2 = 1$. This corresponds with the original formulation of coloring.

Definition 5.4. For an abelian group G, a knot-projection K is **G-colorable** if it has a non-trivial coloring over G.

The following lemma is a generalization of the earlier Lemma 3.6. It assumes a finite abelian group as otherwise it would not make sense.

Lemma 5.5. *For a finite abelian group G, a knot-projection K is G-colorable if and only if $|C_G(K)| > |G|$.*

Proof. There is a trivial coloring for each element of G so this lemma follows naturally from the definition.

□

With these basic ideas we can now prove the main theorem relating the colorings of equivalent knot-projections which establishes the invariance of coloring.

Theorem 5.6. *Let G be an abelian group. For two equivalent knot projections K and K', there is a bijection between their colorings over G.*

Proof. As K and K' are equivalent, there is a number $n \in \mathbb{N}$ of Reidemeister moves to take K to K'. These moves create a sequence of intermediate knot projections, $K = K_0 \leftrightarrow K_1 \leftrightarrow \cdots \leftrightarrow K_n = K'$, such that each K_i and K_{i+1} only differ by one Reidemeister move. By establishing bijections $h_i : C_G(K_i) \leftrightarrow C_G(K_{i+1})$, the theorem will be proven as $h = h_{n-1} \circ h_{n-2} \circ \cdots \circ h_0$ will be the desired bijection.

Claim. *If knot projections K and K' differ by exactly one Reidemeister move, then there is a bijection between their colorings, $C_G(K)$ and $C_G(K')$.*

Proof. The proof idea is to establish an invertible injection between the two sets. For each direction of each Reidemeister move, an injective map will take the colorings on K to the colorings on K'. As the moves are invertible, the map will be also. This will establish the bijection.

For the local region along a Reidemeister move, denote H to be the restriction of K to the region and likewise for H' with relation to K'. Denote K^- to be the common structure of K and K'. K^- is not a knot, but will be called a *partial knot*. A coloring on a partial knot will have the same requirements at any crossing, but for where the partial knot "ends" there are no requirements.

For each Reidemeister move, a coloring on K will restrict to a coloring on K^-. It is enough to show that there is exactly one way to extend the coloring on K^- to a coloring in K'. We avoid the full analysis of all six cases and instead mention the main ideas throughout the cases and apply them directly to two such cases.

To extend a coloring on K^- to a coloring on K' it must be that the "ends" of K^- connect to H' in the same way they connect to H. Thus, the "external" arcs (the ones that connect to K^-) of H' and H must agree in color. The crossing conditions of H show that such an assignment of colors to H' is well-defined. After this assignment occurs, it must be that the rest of the coloring in H' is uniquely determined. The crossing conditions of H and H' are used to prove this. By showing the coloring on H' is unique, the map is injective. To show that these ideas work, we present two of the six cases in detail.

Take the case of the first Reidemeister move, where we take Figure 1a to Figure 1b. The new crossing condition was derived explicitly to show this case preserved colorings. We do not explore it further and instead turn to a more interesting case.

Figure 3 shows the third Reidemeister move. Let the a_i's and the b_i's denote the colors of the arcs instead of the arcs themselves. The arcs a_6 and b_6 are internal as they do not connect to K^-, while the rest are external. Consider the direction taking H, the partial knot in Figure 3a, to H', the partial knot in Figure 3b. Applying the first idea from above shows that extending K^- to K' requires that $b_i = a_i$ for $1 \leq i \leq 5$ for the extension to be well-defined. The crossing conditions on H and H' are expressed in the following linear systems

$$(5.7) \qquad M_H = \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix} \qquad M_{H'} = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 & 0 & 1 \end{bmatrix}$$

As K is a coloring, $\vec{a} = (a_1, \ldots, a_6)^T$ must have $M_H\vec{a} = \vec{0}$, and similarly, $\vec{b} = (b_1, \ldots, b_6)^T$ has $M_{H'}\vec{b} = \vec{0}$ for any valid coloring on K'. As $b_1, \ldots, b_5$ are uniquely determined so far, it remains to show that b_6 is uniquely determined by the requirement $M_{H'}\vec{b} = \vec{0}$. The equation $b_6 = b_4 - 2b_3 = b_1 - 2b_2$ defines the restrictions on b_6 in H'. As $b_i = a_i$ for $0 \leq i \leq 5$, it suffices to show the equation $a_1 - 2a_2 + 2a_3 - a_4 = 0$ holds to conclude b_6 is uniquely determined. This is equivalent to saying the vector $\begin{bmatrix} 1 & -2 & 2 & -1 & 0 & 0 \end{bmatrix}$ is in the rowspace of M_H. The linear combination $\begin{bmatrix} 1 & -2 & -1 \end{bmatrix} M_H$ shows the vector is in the rowspace and so b_6 is uniquely determined. Thus a coloring on K^- derived from K can be extended to a coloring K' in exactly one way.

The other cases are similarly shown using the ideas from above.

□

The proof of the claim establishes that each pair of knot-projections that are one Reidemeister move apart have a bijection between their colorings. As mentioned above, this gives a composition of bijections to establish the bijection between the colorings of any two equivalent knot-projections.

□

Applying Theorem 5.6 to finite abelian groups gives the following corollary which states the knot-invariance of coloring.

Corollary 5.8. *Fix a finite abelian group G. Then the number of colorings is a knot-invariant. G-colorability is also an knot-invariant.*

Along with Section 6, this corollary is one of the main results of the paper. As $\mathbb{Z}_3$ is a finite abelian group the claims from Section 3 that the knot-invariance of the dimension of the colorings are now justified.

Further, we can apply this generalization to yield new results. Consider the 5-foil knot of Figure 9. Coloring over $\mathbb{Z}_3$ failed to show that the 5-foil differs from the unknot. However, if we color over $\mathbb{Z}_5$ it can be verified that there is the non-trivial coloring $\begin{bmatrix} 4 & 3 & 2 & 1 & 0 \end{bmatrix}^T$. As the unknot never has a non-trivial coloring, this proves that the 5-foil differs from the unknot. Returning to the $\mathbb{Z}_3$ case shows the 5-foil also differs from the trefoil knot and the connected-sum of two trefoil knots.

6. The Power of Prime Fields

In the previous section the idea of coloring was expanded to any abelian group. We saw that the number of colorings is a knot-invariant for finite abelian groups. However, there are many different types of abelian groups with a variety of different structures. The prime fields $\mathbb{Z}_p$, for p prime, are easy to compute with when using vector spaces, as seen in Section 3 with the specific case of $\mathbb{Z}_3$. In this section we show that the prime fields are in some sense the only groups that we need to color over so the results of this paper are more easily applied to actual knots.

To begin, we need that the set of colorings $C_G(K)$ has more structure than just a set. In particular, we have the following lemma.

Lemma 6.1. *For a knot-projection K with n arcs and an abelian group G, $C_G(K)$ is isomorphic to a subgroup of the product group G^n. Furthermore, $C_G(K)$ is the kernel of the group homomorphism $\varphi : G^n \to G^n$ defined by $\varphi(\vec{g}) = M_K\vec{g}$, where M_K is the coloring matrix of K.*

Proof. For the arcs $A(K) = \{a_1, \dots, a_n\}$ we can define $C = \{(f(a_1), \dots, f(a_n)) \in G^n : f \in C_G(K)\} \subseteq G^n$. C is a group as the crossing conditions are linear. Specifically, the identity axiom holds as the zero coloring is in $C_G(K)$. Inverses exist because for any crossing with over-arc a_i and under-arcs a_j and a_k, a coloring $f \in C_G(K)$ has $2f(a_i) = f(a_j) + f(a_k)$ and so implies $2f(a_i)^{-1} = -2f(a_i) = -f(a_j) - f(a_k) = f(a_j)^{-1} + f(a_k)^{-1}$ and thus $-f \in C_G(K)$. If $f, h \in C_G(K)$ then for a crossing with over-arc a_i and under-arcs a_j and a_k, $2(f(a_i) + h(a_i)) = (f(a_j) + h(a_j)) + (f(a_k) + h(a_k))$. Thus $f + h \in C_G(K)$. Associativity is inherited, so we then have that C is a subgroup of G^n.

The second part of the claim follows from the definition of the coloring matrix and its relation to the set of colorings. The notion $M_K\vec{g}$ is well-defined as M_K is an integer matrix. □

The addition structure on the set of colorings gives the power to prove the next result, which is the inspiration for this entire section.

Theorem 6.2. *For a knot-projection K with n arcs and a finite abelian group G with subgroup H, $|C_G(K)| \leq |C_H(K)| \cdot |C_{G/H}(K)|$.*

Proof. Consider the canonical homomorphism $\varphi : G \to G/H$. By Lemma 6.1, we can then use φ to construct the canonical group homomorphism $\pi : C_G \to (G/H)^n$. Notice $\ker \pi = C_G \cap H^n = C_H$. By construction of M_k, any $c \in C_G(K)$ has $M_K c = \vec{0}_{G^n}$. As linear equations commute over homomorphisms, $M_K \pi(c) = \pi(M_K c) = \pi(\vec{0}_{G^n}) = \vec{0}_{(G/H)^n}$. Thus, $\pi(c) \in (G/H)^n$ satisfies the crossing conditions so $\pi(c) \in C_{G/H}(K)$. Thus $\operatorname{im} \pi \subseteq C_{G/H}(K)$. The claim follows from $|C_G(K)| = |\ker \pi| \cdot |\operatorname{im} \pi| \leq |C_H(K)| \cdot |C_{G/H}(K)|$. □

This theorem allows the knot-invariant property of G-colorability to be related to subgroups.

Corollary 6.3. *For a knot-projection K and a finite abelian group G with subgroup H, K is G-colorable if and only if it is H-colorable or it is G/H-colorable.*

Proof. By Theorem 6.2, $|C_G(K)| \leq |C_H(K)| \cdot |C_{G/H}(K)|$. As $|G| = |H| \cdot |G/H|$, it must that $|C_G(K)| > |G|$ if and only if at least of $|H| > |C_H(K)|$ or $|G/H| > |C_{G/H}(K)|$. By Lemma 5.5, K is G-colorable if and only if $|C_G(K)| > |G|$. Combining these statements gives the corollary. □

Recursively applying Corollary 6.3 gives the main corollary relating G-colorability to $\mathbb{Z}_p$-colorability, thus showing the power of prime fields in this notion of coloring.

Corollary 6.4. *For a finite abelian group G, a knot-projection K is G-colorable if and only if there is a prime p where K is $\mathbb{Z}_p$-colorable and p divides $|G|$.*

Corollary 5.8 establishes that G-colorability is a knot-invariant. Therefore, Corollary 6.4 shows that two knots can be differentiated by the property of G-colorability if and only if they can be differentiated by $\mathbb{Z}_p$-colorability for some p. As Section 3 can be naturally generalized to work with any field $\mathbb{Z}_p$, this gives a framework for differentiating knots that can be done algorithmically as vector spaces are well-understood by computers.

A natural question to ask is whether Theorem 6.2 can be strengthened so that equality always holds or holds for some class of interesting groups. If equality were to hold then the number of colorings of a knot over a group G is fully determined by the colorings over smaller groups, and therefore by the groups $\mathbb{Z}_p$. Therefore, when attempting to distinguish knots via the number of colorings, as opposed to just G-colorability, the groups $\mathbb{Z}_p$ are the only groups that need to be examined. The next result shows that equality does hold for the groups $\mathbb{Z}_n$.

Theorem 6.5. *For a knot-projection K with n arcs and group $\mathbb{Z}_{rs}$, $|C_{\mathbb{Z}_{rs}}(K)| = |C_{\mathbb{Z}_r}(K)| \cdot |C_{\mathbb{Z}_s}(K)|$.*

Proof. Let M_K be the coloring matrix of K. Denote $r\mathbb{Z}_{rs}^n = \{r\vec{x} : \vec{x} \in \mathbb{Z}_{rs}^n\}$ and $M_K\mathbb{Z}_{rs}^n = \{M\vec{x} : \vec{x} \in \mathbb{Z}_{rs}^n\}$. As M_K is a $n \times n$ matrix we get that both $r\mathbb{Z}_{rs}^n$ and $M_K\mathbb{Z}_{rs}^n$ are both subgroups of $\mathbb{Z}_{rs}^n$. Define $M_K r\mathbb{Z}_{rs}^n$ analogously. Consider the following commutative diagram

(6.6)
$$\begin{array}{ccc} \mathbb{Z}_{rs}^n & \xrightarrow{\varphi_1(\vec{x})=r\vec{x}} & r\mathbb{Z}_{rs}^n \\ {\scriptstyle \psi_2(\vec{x})=M_K\vec{x}}\downarrow & & \downarrow{\scriptstyle \psi_1(\vec{x})=M_K\vec{x}} \\ M_K\mathbb{Z}_{rs}^n & \xrightarrow{\varphi_2(\vec{x})=r\vec{x}} & M_K r\mathbb{Z}_{rs}^n \end{array}$$

By inspection, we see that $\ker\varphi_1 = s\mathbb{Z}_{rs}^n$, $\ker\psi_2 = C_{\mathbb{Z}_{rs}}$, $\ker\psi_1 = C_{r\mathbb{Z}_{rs}}$. It is clear that $\ker\varphi_2 = (M_K\mathbb{Z}_{rs}^n) \cap (s\mathbb{Z}_{rs}^n)$, but a stronger statement is needed. To get there we prove the following claim, which rests on the First, Second and Third Group Isomorphism Theorems [1].

Claim. *$M_K r\mathbb{Z}_{rs}^n$ is isomorphic to $\mathbb{Z}_{rs}^n/(C_{\mathbb{Z}_{rs}} + s\mathbb{Z}_{rs}^n)$.*

Proof. We analyze the homomorphism $\pi = \psi_1\varphi_1 = \varphi_2\psi_2$ which has $\pi : \mathbb{Z}_{rs}^n \to M_K r\mathbb{Z}_{rs}^n$. For $\vec{x} \in \mathbb{Z}_{rs}^n$, if $\pi(\vec{x}) = \vec{0}_{M_K r\mathbb{Z}_{rs}^n}$ then $M_K r\vec{x} = \vec{0}_{\mathbb{Z}_{rs}^n}$. Thus $r\vec{x} = \vec{c} \in C_{\mathbb{Z}_{rs}}$. Thus r divides the entries of $\vec{c}$ so take $\vec{y} = \vec{c}/r$ and $\vec{z} = \vec{x} - \vec{y}$. As $r(\vec{x} - \vec{y}) = \vec{0}$. $\vec{z} \in s\mathbb{Z}_{rs}^n$ and $\vec{x} = \vec{y} + \vec{z}$. Therefore $\ker\pi \subseteq C_{\mathbb{Z}_{rs}} + s\mathbb{Z}_{rs}^n$. The other direction can run this argument in reverse, so $\ker\pi = C_{\mathbb{Z}_{rs}} + s\mathbb{Z}_{rs}^n$. As π is surjective by construction, the First Isomorphism Theorem gives the claim.

□

There is a natural homomorphism between $\mathbb{Z}_{rs}^n/C_{\mathbb{Z}_{rs}}$ and $\mathbb{Z}_{rs}^n/(C_{\mathbb{Z}_{rs}} + s\mathbb{Z}_{rs}^n)$ with kernel $(C_{\mathbb{Z}_{rs}} + s\mathbb{Z}_{rs}^n)/C_{\mathbb{Z}_{rs}}$ as given by the Third Isomorphism Theorem. By the Second Isomorphism Theorem, $(C_{\mathbb{Z}_{rs}} + s\mathbb{Z}_{rs}^n)/C_{\mathbb{Z}_{rs}}$ is isomorphic to $s\mathbb{Z}_{rs}^n/(C_{\mathbb{Z}_{rs}} \cap s\mathbb{Z}_{rs}^n)$. But $C_{\mathbb{Z}_{rs}} \cap s\mathbb{Z}_{rs}^n = C_{s\mathbb{Z}_{rs}}$ and so the kernel of the map between $\mathbb{Z}_{rs}^n/C_{\mathbb{Z}_{rs}}$ and $\mathbb{Z}_{rs}^n/(C_{\mathbb{Z}_{rs}} + s\mathbb{Z}_{rs}^n)$ is of order $|C_{s\mathbb{Z}_{rs}}|$.

Thus $|\mathbb{Z}_{rs}^n/C_{\mathbb{Z}_{rs}}| = |C_{s\mathbb{Z}_{rs}}| \cdot |\mathbb{Z}_{rs}^n/(C_{\mathbb{Z}_{rs}} + s\mathbb{Z}_{rs}^n)|$. Applying the First Isomorphism Theorem gives $M_K\mathbb{Z}_{rs}^n \cong \mathbb{Z}_{rs}^n/C_{\mathbb{Z}_{rs}}$ and so $|M_K\mathbb{Z}_{rs}^n| = |s\mathbb{Z}_{rs}^n/C_{s\mathbb{Z}_{rs}}| \cdot |rM_K\mathbb{Z}_{rs}^n|$. Thus $|\ker\varphi_2| = |\mathbb{Z}_{rs}^n/C_{s\mathbb{Z}_{rs}}| = |s\mathbb{Z}_{rs}^n|/|C_{s\mathbb{Z}_{rs}}|$. This is the stronger statement we needed.

We can read off the diagram that $|\ker\psi_1| \cdot |\ker\varphi_1| = |\ker\psi_2| \cdot |\ker\varphi_2|$ or, that $|C_{r\mathbb{Z}_{rs}}| \cdot |s\mathbb{Z}_{rs}^n| = |s\mathbb{Z}_{rs}^n|/|C_{s\mathbb{Z}_{rs}}| \cdot |C_{\mathbb{Z}_{rs}}|$. As $r\mathbb{Z}_{rs} \cong \mathbb{Z}_s$ and $s\mathbb{Z}_{rs} \cong \mathbb{Z}_r$ this gives the theorem.

□

Just as Theorem 6.2 gives Corollary 6.3 so does Theorem 6.5 give the following corollary asserting a more powerful statement about the ability of prime fields to differentiate knots.

Corollary 6.7. *Two knot-projections K and K' can be differentiated by counting the number of colorings over $\mathbb{Z}_n$ if and only if they can be differentiated by counting the number of colorings over $\mathbb{Z}_p$ for some prime p dividing n.*

This last corollary is very informative and it would be convenient if it held for all finite abelian groups G. We are not sure. The only property used of $\mathbb{Z}_n$ was that for any subgroup H of $\mathbb{Z}_n$, $\mathbb{Z}_n/H$ is a subgroup of $\mathbb{Z}_n$. Proving the theorem for any group with such a property seems notationally difficult, so we leave it with the following conjecture.

Conjecture 6.8. *Fix a knot-projection K. For a group G with subgroup H, if G/H is isomorphic to a subgroup of G then $|C_G(K)| = |C_H(K)| \cdot |C_{G/H}(K)|$.*

7. Further Directions

We showed that the number of colorings of a knot over a finite abelian group is a knot-invariant. For full generality, we would like to take this invariant in two directions: first to infinite fields where dimension would be the knot-invariant, and second to arbitrary finite groups. These generalizations would be enhanced by stronger theorems, such as proving that the bijection of Theorem 5.6 is actually a group isomorphism. Section 6 seems to break down on non-abelian groups and ideally this relationship could be better explored. Further, Conjecture 6.8 needs to be proven or refuted. Aside from further generalizations of our results, another direction would be to apply those results to distinguish other examples of inequivalent knots, as well as to explore instances when our general notion of coloring over abelian groups, or over prime fields, fails to distinguish inequivalent knots.

8. Division Of Labor

Matthew: Abstract, Introduction, Colorability
Justin: Representing Knot Projections with Linear Systems, Coloring the Connected Sum
Michael: Generalizations of Coloring to Groups, The Power of Prime Fields

References

1. Michael Artin, Algebra, Prentice Hall, 1991.
2. Eric W. Weisstein, Mathworld: The web's most extensive mathematics resource, [Online; accessed 14-March-2008].
3. Wikipedia, Unknot — Wikipedia, the free encyclopedia, [Online; accessed 14-March-2008].